10 Confidence intervals

10.1 Estimation uncertainty

An estimator provides an approximation of an unknown population parameter as a single real number or vector, which we call a **point estimate**. For instance, when we estimate the linear relationship between wage, education, and gender using an OLS, we obtain a specific set of coefficients:

```
cps = read.csv("cps.csv")
lm(wage ~ education + female, data = cps) |> coef()
```

```
(Intercept)   education      female 
 -14.081788    2.958174   -7.533067 
```

However, the point estimate $\widehat{\beta}_j$ alone does not reflect how close or far the estimate might be from the true population parameter β_j. It doesn't capture estimation uncertainty. This inherent uncertainty arises because point estimates are based on a finite sample, which may vary from sample to sample.

Larger samples tend to give more accurate OLS estimates as OLS is unbiased and consistent under assumptions (A1)–(A4). However, we work with fixed, finite samples in practice.

Confidence intervals address this limitation by providing a range of values likely to contain the true population parameter. By constructing an interval around our point estimate that contains the true parameter with a specified probability (e.g., 95% confidence level), we can express the uncertainty more clearly.

In this section, we will introduce **interval estimates**, commonly referred to as **confidence intervals**. To construct a confidence interval for an OLS coefficient $\widehat{\beta}_j$, we need two components: a **standard error** (an estimate of the standard deviation of the estimator) and information about the distribution of $\widehat{\beta}_j$.

10.2 Gaussian distribution

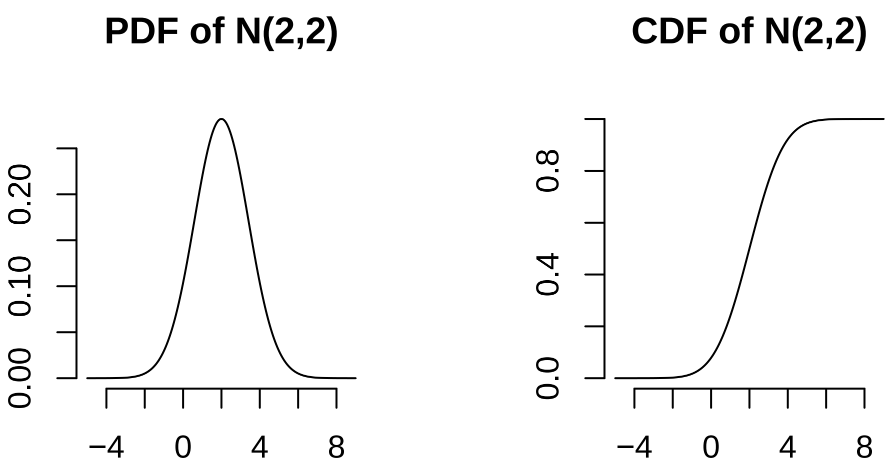
The **Gaussian distribution**, also known as the **normal distribution**, is a fundamental concept in statistics. We often use these terms interchangeably: a random variable Z is said to follow a Gaussian or normal distribution if it has the following probability density function (PDF) with a given mean μ and variance σ^2:

$$f(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{(u-\mu)^2}{2\sigma^2}\Big).$$

Formally, we denote this as $Z \sim \mathcal{N}(\mu, \sigma^2)$, meaning that Z is normally distributed with mean μ and variance σ^2.

- Mean: $E[Z] = \mu$
- Variance: $Var(Z) = \sigma^2$
- Skewness: $skew(Z) = 0$
- Kurtosis: $kurt(Z) = 3$

```
par(mfrow=c(1,2), bty="n", lwd=1)
x = seq(-5,9,0.01) # define grid for x-axis for the plot
plot(x, dnorm(x, mean = 2, sd = sqrt(2)), type="l", main="PDF of N(2,2)", ylab="", xlab="")
plot(x, pnorm(x, mean = 2, sd = sqrt(2)), type="l", main="CDF of N(2,2)", ylab="", xlab="")
```



Use the R functions `dnorm` to calculate normal PDF values and `pnorm` for normal CDF values.

The Gaussian distribution with mean 0 and variance 1 is called the **standard normal distribution**. It has the PDF

$$\phi(u) = \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{u^2}{2}\Big)$$

and CDF

$$\Phi(a) = \int_{-\infty}^{a} \phi(u) \, \mathrm{d}u.$$

$\mathcal{N}(0,1)$ is symmetric around zero:

$$\phi(u) = \phi(-u), \quad \Phi(a) = 1 - \Phi(-a)$$

Standardizing: If $Z \sim \mathcal{N}(\mu, \sigma^2)$, then

$$\frac{Z - \mu}{\sigma} \sim \mathcal{N}(0,1),$$

and the CDF of Z is $\Phi((Z-\mu)/\sigma)$.

Linear combinations of normally distributed variables are normal: If $Y_1, \ldots, Y_n$ are normally distributed and $c_1, \ldots, c_n \in \mathbb{R}$, then $\sum_{j=1}^{n} c_j Y_j$ is normally distributed.

10.2.1 Multivariate Gaussian distribution

Let $Z_1, \ldots, Z_k$ be independent $\mathcal{N}(0,1)$ random variables. Then, the k-vector $\boldsymbol{Z} = (Z_1, \ldots, Z_k)'$ has the **multivariate standard normal distribution**, written $\boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_k)$. Its joint density is

$$f(\boldsymbol{u}) = \frac{1}{(2\pi)^{k/2}} \exp\left(-\frac{\boldsymbol{u}'\boldsymbol{u}}{2}\right).$$

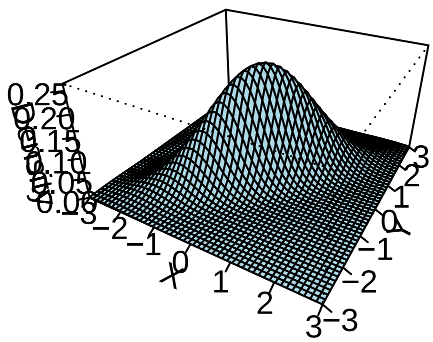
If $\boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_k)$ and $\boldsymbol{Z}^* = \boldsymbol{\mu} + \boldsymbol{B}\boldsymbol{Z}$ for a $q \times 1$ vector $\boldsymbol{\mu}$ and a $q \times k$ matrix $\boldsymbol{B}$, then $\boldsymbol{Z}^*$ has a **multivariate normal distribution** with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma} = \boldsymbol{B}\boldsymbol{B}'$, written $\boldsymbol{Z}^* \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. The k-variate PDF of $\boldsymbol{Z}^*$ is

$$f(\boldsymbol{u}) = \frac{1}{(2\pi)^{k/2}(\det(\boldsymbol{\Sigma}))^{1/2}} \exp\Big(-\frac{1}{2}(\boldsymbol{u} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{u} - \boldsymbol{\mu})\Big).$$

The mean vector and covariance matrix are

$$E[\boldsymbol{Z}^*] = \boldsymbol{\mu}, \quad Var(\boldsymbol{Z}^*) = \boldsymbol{\Sigma}.$$

3D Bivariate Normal Distribution Density



The 3d plot shows the bivariate normal PDF with parameters

$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0.8 \\ 0.8 & 1 \end{pmatrix}.$$

10.2.2 Chi-squared distribution

Let $Z_1, \dots, Z_m$ be independent $\mathcal{N}(0,1)$ random variables. Then, the random variable

$$Y = \sum_{i=1}^{m} Z_i^2$$

is **chi-squared distributed** with parameter m, written $Y \sim \chi_m^2$.

The parameter m is called the degrees of freedom.

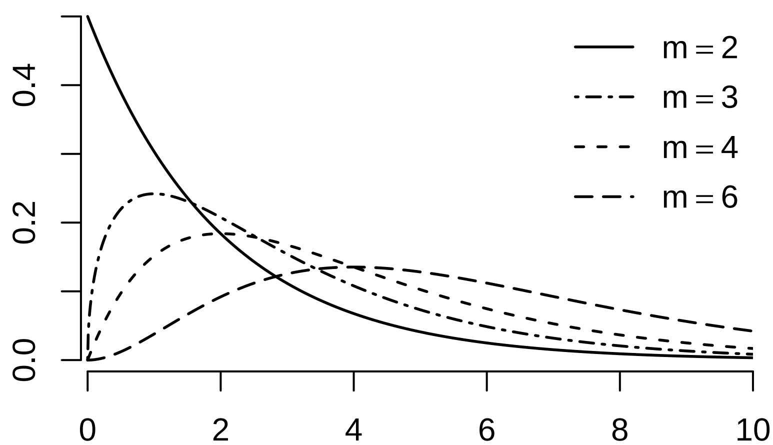


Figure 10.1: PDF of the χ^2-distribution

- Mean: $E[Y] = m$
- Variance: $Var(Y) = 2m$
- Skewness: $skew(Y) = \sqrt{8/m}$
- Kurtosis: $kurt(Y) = 3 + 12/m$

10.2.3 Student t-distribution

If $Z \sim \mathcal{N}(0,1)$ and $Q \sim \chi^2_m$, and Z and Q are independent, then

$$Y = \frac{Z}{\sqrt{Q/m}}$$

is t-**distributed** with parameter m degrees of freedom, written $Y \sim t_m$.

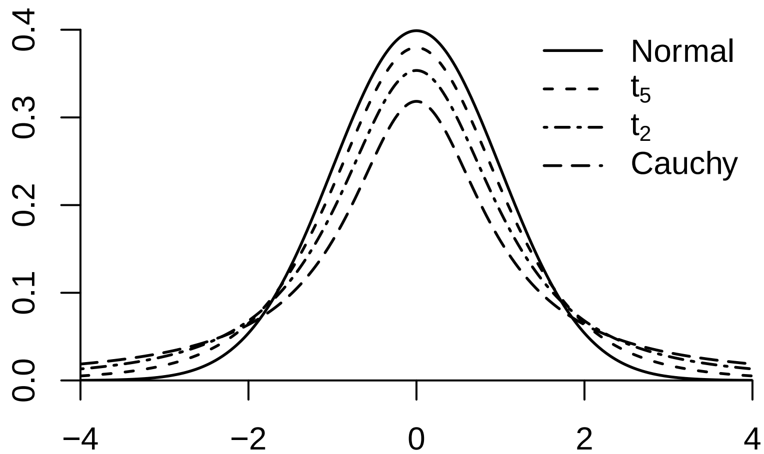


Figure 10.2: PDFs of the Student t-distribution

The t-distribution with $m = 1$ is also called **Cauchy distribution**. The t-distributions with 1, 2, 3, and 4 degrees of freedom are heavy-tailed distributions. If $m \to \infty$ then $t_m \to \mathcal{N}(0,1)$

- Mean: $E[Y] = 0$ if $m \geq 2$
- Variance: $Var(Y) = \frac{m}{m-2}$ if $m \geq 3$
- Skewness: $skew(Y) = 0$ if $m \geq 4$
- Kurtosis: $kurt(Y) = 3 + 6/(m-4)$ if $m \geq 5$

The kurtosis is infinite for $m \leq 4$, the skewness is undefined for $m \leq 3$, the variance is infinite for $m \leq 2$, and the mean is undefined for $m = 1$.

10.3 Classical Gaussian regression model

Let's revisit the linear regression model:

$$Y_i = \boldsymbol{X}_i'\boldsymbol{\beta} + u_i, \quad i = 1, \dots, n. \tag{10.1}$$

Under assumptions (A1)–(A4), the distributional restrictions on the error term are relatively mild:

1) The error terms are i.i.d. but can have different conditional variances depending on the values of the regressors (heteroskedasticity):

$$Var(u_i|\boldsymbol{X}_i) = \sigma^2(\boldsymbol{X}_i) = \sigma_i^2.$$

For example, in a regression of `wage` on `female`, the error variances for women may differ from those for men.

2) The error term can follow any distribution, provided that the fourth moment (the kurtosis) is finite. This excludes heavy-tailed distributions.

In standard introductory textbooks, two additional assumptions are often made to further restrict the properties mentioned above. It is beneficial to first study the estimation uncertainty under this simplified setting.

Classical Gaussian regression model

In addition to the linear regression model in Equation 10.1 with assumptions (A1)–(A4), we introduce two more assumptions:

- (A5) **Homoskedasticity**: The error terms have constant variance across all observations, i.e.,

$$Var(u_i|\boldsymbol{X}_i) = \sigma_i^2 = \sigma^2 \quad \text{for all } i = 1, \dots, n.$$

- (A6) **Normality**: The error terms are normally distributed conditional on the regressors, i.e.,

$$u_i|\boldsymbol{X}_i \sim \mathcal{N}(0, \sigma_i^2).$$

(A5)-(A6) combined can be expressed as:

$$u_i|\boldsymbol{X}_i \sim \mathcal{N}(0, \sigma^2) \quad \text{for all } i = 1, \dots, n.$$

The notation $u_i|\boldsymbol{X}_i \sim \mathcal{N}(0, \sigma^2)$ means that the conditional distribution of u_i conditional on $\boldsymbol{X}_i$ is $N(0, \sigma^2)$. The PDF of $u_i|\boldsymbol{X}_i$ is

$$f(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{u^2}{2\sigma^2}\Big).$$

Distribution of the OLS coefficients

Conditional on $\boldsymbol{X}$, the OLS coefficient vector is a linear combination of the error term:

$$\begin{aligned}\widehat{\boldsymbol{\beta}} &= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y} \\ &= \boldsymbol{\beta} + (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{u}.\end{aligned}$$

Consequently, under (A6), the OLS estimator follows a k-variate normal distribution, conditionally on $\boldsymbol{X}$.

Recall that the mean is $E[\widehat{\boldsymbol{\beta}}|\boldsymbol{X}] = \boldsymbol{\beta}$ and the covariance matrix is

$$Var(\widehat{\boldsymbol{\beta}}|\boldsymbol{X}) = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{D}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}.$$

Under homoskedasticity (A5), we have $\boldsymbol{D} = \sigma^2\boldsymbol{I}_n$, so the covariance matrix simplifies to

$$Var(\widehat{\boldsymbol{\beta}}|\boldsymbol{X}) = \sigma^2(\boldsymbol{X}'\boldsymbol{X})^{-1}.$$

Therefore,

$$\widehat{\boldsymbol{\beta}}|\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\beta}, \sigma^2(\boldsymbol{X}'\boldsymbol{X})^{-1}).$$

The variance of the j-th OLS coefficient is

$$Var(\widehat{\beta}_j|\boldsymbol{X}) = \sigma^2[(\boldsymbol{X}'\boldsymbol{X})^{-1}]_{jj},$$

where $[(\boldsymbol{X}'\boldsymbol{X})^{-1}]_{jj}$ indicates the j-th diagonal element of the matrix $(\boldsymbol{X}'\boldsymbol{X})^{-1}$. The standard deviation is:

$$sd(\widehat{\beta}_j|\boldsymbol{X}) = \sqrt{\sigma^2[(\boldsymbol{X}'\boldsymbol{X})^{-1}]_{jj}}.$$

Therefore, the standardized OLS coefficient has a standard normal distribution:

$$Z_j := \frac{\widehat{\beta}_j - \beta_j}{sd(\widehat{\beta}_j|\boldsymbol{X})} \sim \mathcal{N}(0,1). \tag{10.2}$$

10.4 Confidence interval: known variance

One of the most common methods of incorporating estimation uncertainty into estimation results is through **interval estimates**, often referred to as **confidence intervals**.

A confidence interval is a range of values that is likely to contain the true population parameter with a specified **confidence level** or **coverage probability**, often expressed as a percentage (e.g., 95%). For example, a 95% confidence interval suggests that, across many repeated samples, approximately 95% of the intervals constructed from those samples would contain the true population parameter.

A symmetric confidence interval for β_j with confidence level $1-\alpha$ is an interval

$$I_{1-\alpha} = [\hat{\beta}_j - c_{1-\alpha}; \hat{\beta}_j + c_{1-\alpha}]$$

with the property that

$$P(\beta_j \in I_{1-\alpha}) = 1 - \alpha. \tag{10.3}$$

Common coverage probabilities are 0.95, 0.99, and 0.90.

Note that $I_{1-\alpha}$ is random and β_j is fixed but unknown. Therefore, the coverage probability is the probability that this random interval $I_{1-\alpha}$ contains the true parameter.

A more precise interpretation of a confidence interval is:

If we were to repeat the sampling process and construct confidence intervals for each sample, $1-\alpha$ of those intervals would contain the true population parameter.

It is essential to understand that the confidence interval reflects the reliability of the method, not the probability of the true parameter falling within a particular interval. The interval itself is random – it varies with each sample – but the population parameter is fixed and unknown.

Thus, it is incorrect to interpret a specific confidence interval as having a 95% probability of containing the true value. Instead, the correct interpretation is that the method used to calculate the interval has a 95% success rate across many samples.

The width of the interval

The OLS coefficient $\hat{\beta}_j$ is in the center of $I_{1-\alpha}$. Let's solve for $c_{1-\alpha}$ to get the width of the confidence interval.

The event $\{\beta_j \in I_{1-\alpha}\}$ can be rearranged as

$$\begin{aligned}
& \beta_j \in I_{1-\alpha} \\
\Leftrightarrow \quad & \hat{\beta}_j - c_{1-\alpha} \leq \beta_j \leq \hat{\beta}_j + c_{1-\alpha} \\
\Leftrightarrow \quad & -c_{1-\alpha} \leq \beta_j - \hat{\beta}_j \leq c_{1-\alpha} \\
\Leftrightarrow \quad & c_{1-\alpha} \geq \hat{\beta}_j - \beta_j \geq -c_{1-\alpha} \\
\Leftrightarrow \quad & \frac{c_{1-\alpha}}{sd(\hat{\beta}_j|\boldsymbol{X})} \geq Z_j \geq -\frac{c_{1-\alpha}}{sd(\hat{\beta}_j|\boldsymbol{X})}
\end{aligned}$$

with Z_j defined in Equation 10.2. Hence, Equation 10.3 becomes

$$P\left(\frac{-c_{1-\alpha}}{sd(\hat{\beta}_j|\boldsymbol{X})} \leq Z_j \leq \frac{c_{1-\alpha}}{sd(\hat{\beta}_j|\boldsymbol{X})}\right) = 1 - \alpha. \tag{10.4}$$

Since Z_j is standard normal by Equation 10.2, we have

$$\begin{aligned}
&P\left(\frac{-c_{1-\alpha}}{sd(\widehat{\beta}_j|\boldsymbol{X})} \leq Z_j \leq \frac{c_{1-\alpha}}{sd(\widehat{\beta}_j|\boldsymbol{X})}\right) \\
&= \Phi\left(\frac{c_{1-\alpha}}{sd(\widehat{\beta}_j|\boldsymbol{X})}\right) - \Phi\left(\frac{-c_{1-\alpha}}{sd(\widehat{\beta}_j|\boldsymbol{X})}\right) \\
&= \Phi\left(\frac{c_{1-\alpha}}{sd(\widehat{\beta}_j|\boldsymbol{X})}\right) - \left(1 - \Phi\left(\frac{c_{1-\alpha}}{sd(\widehat{\beta}_j|\boldsymbol{X})}\right)\right) \\
&= 2\Phi\left(\frac{c_{1-\alpha}}{sd(\widehat{\beta}_j|\boldsymbol{X})}\right) - 1.
\end{aligned}$$

With Equation 10.4, we get

$$1 - \alpha = 2\Phi\left(\frac{c_{1-\alpha}}{sd(\widehat{\beta}_j|\boldsymbol{X})}\right) - 1.$$

Let's add 1 and divide by 2:

$$1 - \frac{\alpha}{2} = \Phi\left(\frac{c_{1-\alpha}}{sd(\widehat{\beta}_j|\boldsymbol{X})}\right), \tag{10.5}$$

where $(2-\alpha)/2 = 1 - \alpha/2$.

The value $z_{(p)}$ is the **p-quantile** of $\mathcal{N}(0,1)$ if $\Phi(z_{(p)}) = p$. We write $\Phi^{-1}(p) = z_{(p)}$, where the quantile function Φ^{-1} is the inverse function of the CDF Φ with $\Phi(\Phi^{-1}(p)) = p$ and $\Phi^{-1}(\Phi^{-1}(z)) = z$.

Then, applying the quantile function Φ^{-1} to Equation 10.5 gives:

$$\begin{aligned}
\Leftrightarrow \qquad & \Phi^{-1}\left(1 - \frac{\alpha}{2}\right) = \frac{c_{1-\alpha}}{sd(\widehat{\beta}_j|\boldsymbol{X})} \\
\Leftrightarrow \qquad & z_{(1-\frac{\alpha}{2})} = \frac{c_{1-\alpha}}{sd(\widehat{\beta}_j|\boldsymbol{X})} \\
\Leftrightarrow \qquad & z_{(1-\frac{\alpha}{2})} \cdot sd(\widehat{\beta}_j|\boldsymbol{X}) = c_{1-\alpha},
\end{aligned}$$

where $z_{(1-\frac{\alpha}{2})}$ is the $1-\alpha/2$-quantile of $\mathcal{N}(0,1)$. The solution for the confidence interval is:

$$I_{1-\alpha} = \left[\widehat{\beta}_j - z_{(1-\frac{\alpha}{2})} \cdot sd(\widehat{\beta}_j|\boldsymbol{X});\ \widehat{\beta}_j + z_{(1-\frac{\alpha}{2})} \cdot sd(\widehat{\beta}_j|\boldsymbol{X})\right].$$

Standard normal quantiles can be obtained using the R command `qnorm` or by using statistical tables:

Table 10.1: Some quantiles of the standard normal distribution

0.9	0.95	0.975	0.99	0.995
1.28	1.64	1.96	2.33	2.58

Therefore, 90%, 95%, and 99% confidence intervals for β_j are given by

$$\begin{aligned} I_{0.9} &= [\widehat{\beta}_j - 1.64 \cdot sd(\widehat{\beta}_j|\boldsymbol{X});\ \widehat{\beta}_j + 1.64 \cdot sd(\widehat{\beta}_j|\boldsymbol{X})] \\ I_{0.95} &= [\widehat{\beta}_j - 1.96 \cdot sd(\widehat{\beta}_j|\boldsymbol{X});\ \widehat{\beta}_j + 1.96 \cdot sd(\widehat{\beta}_j|\boldsymbol{X})] \\ I_{0.99} &= [\widehat{\beta}_j - 2.58 \cdot sd(\widehat{\beta}_j|\boldsymbol{X});\ \widehat{\beta}_j + 2.58 \cdot sd(\widehat{\beta}_j|\boldsymbol{X})] \end{aligned}$$

With probability α, the interval does not cover the true parameter. The smaller we choose α, the more confident we can be that the interval covers the true parameter, but the larger the interval becomes. If we set $\alpha = 0$, the interval would be infinite, providing no useful information.

A certain amount of uncertainty always remains, but we can control it by choosing an appropriate value for α that balances our desired level of confidence with the precision of the estimate. This is why the coverage probability $(1-\alpha)$ is also called the **confidence level**.

Note that this interval is **infeasible** in practice because the conditional standard deviation is unknown:

$$sd(\widehat{\beta}_j|\boldsymbol{X}) = \sqrt{\sigma^2[(\boldsymbol{X}'\boldsymbol{X})^{-1}]_{jj}}.$$

It requires knowledge about the true error variance $Var(u_i|\boldsymbol{X}) = \sigma^2$.

10.5 Classical standard errors

A standard error $se(\widehat{\beta}_j)$ for an estimator $\widehat{\beta}_j$ is an estimator of the standard deviation of the distribution of $\widehat{\beta}_j$.

We say that the standard error is consistent if

$$\frac{se(\widehat{\beta}_j)}{sd(\widehat{\beta}_j|\boldsymbol{X})} \overset{p}{\to} 1. \tag{10.6}$$

This property ensures that, in practice, we can replace the unknown standard deviation with the standard error in confidence intervals.

Under the classical Gaussian regression model, we have

$$sd(\widehat{\beta}_j|\boldsymbol{X}) = \sqrt{\sigma^2[(\boldsymbol{X}'\boldsymbol{X})^{-1}]_{jj}}.$$

Therefore, it is natural to replace the population error variance σ^2 by the adjusted sample variance of the residuals:

$$s_{\widehat{u}}^2 = \frac{1}{n-k}\sum_{i=1}^{n} \hat{u}_i^2 = SER^2.$$

The classical homoskedastic standard errors are:

$$se_{hom}(\hat{\beta}_j) = \sqrt{s_{\widehat{u}}^2[(\boldsymbol{X}'\boldsymbol{X})^{-1}]_{jj}}.$$

The classical homoskedastic covariance matrix estimator for $Var(\widehat{\boldsymbol{\beta}}|\boldsymbol{X})$ is

$$\widehat{\boldsymbol{V}}_{hom} = s_{\widehat{u}}^2(\boldsymbol{X}'\boldsymbol{X})^{-1}$$

```
fit = lm(wage ~ education + female, data = cps)
## classical homoskedastic covariance matrix estimator:
vcov(fit)
```

```
            (Intercept)     education        female
(Intercept)  0.18825476 -0.0127486354 -0.0089269796
education   -0.01274864  0.0009225111 -0.0002278021
female      -0.00892698 -0.0002278021  0.0284200217
```

The classical standard errors are the square roots of the diagonal elements of this matrix:

```
## classical standard errors:
sqrt(diag(vcov(fit)))
```

```
(Intercept)   education      female
 0.43388334  0.03037287  0.16858239
```

These standard errors are also displayed in the second column of a regression output:

```
summary(fit)
```

```

Call:
lm(formula = wage ~ education + female, data = cps)

Residuals:
    Min      1Q  Median      3Q     Max
-45.071  -9.035  -2.973   4.472 244.491
```

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) -14.08179    0.43388  -32.45   <2e-16 ***
education     2.95817    0.03037   97.39   <2e-16 ***
female       -7.53307    0.16858  -44.69   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 18.76 on 50739 degrees of freedom
Multiple R-squared:  0.1797,	Adjusted R-squared:  0.1797 
F-statistic:  5559 on 2 and 50739 DF,  p-value: < 2.2e-16
```

Because $s_{\widehat{u}}^2/\sigma^2 \overset{p}{\rightarrow} 1$, property Equation 10.6 is satisfied and $se_{hom}(\widehat{\beta}_j)$ is a consistent standard error under homoskedasticity.

Note that the main result we used to derive the confidence interval is that the standardized OLS coefficient is standard normal:

$$Z_j := \frac{\widehat{\beta}_j - \beta_j}{sd(\widehat{\beta}_j|\boldsymbol{X})} \sim \mathcal{N}(0,1).$$

If we replace the unknown standard deviation $sd(\widehat{\beta}_j|\boldsymbol{X})$ with the standard error $se_{hom}(\widehat{\beta}_j)$, the distribution changes.

The OLS estimator standardized with the standard error is called **t-statistic**:

$$T_j = \frac{\widehat{\beta}_j - \beta_j}{se_{hom}(\widehat{\beta}_j)} = \frac{sd(\widehat{\beta}_j|\boldsymbol{X})}{se_{hom}(\widehat{\beta}_j)} \frac{\widehat{\beta}_j - \beta_j}{sd(\widehat{\beta}_j|\boldsymbol{X})} = \frac{sd(\widehat{\beta}_j|\boldsymbol{X})}{se_{hom}(\widehat{\beta}_j)} Z_j.$$

The additional factor satisfies

$$\frac{sd(\widehat{\beta}_j|\boldsymbol{X})}{se_{hom}(\widehat{\beta}_j)} = \frac{\sigma}{s_{\widehat{u}}} \sim \sqrt{(n-k)/\chi^2_{n-k}},$$

where χ^2_{n-k} is the chi-squared distribution with $n-k$ degrees of freedom, independent of Z_j.

Therefore, the t-statistic is t-distributed:

$$T_j = \frac{\widehat{\beta}_j - \beta_j}{se_{hom}(\widehat{\beta}_j)} = \frac{\sigma}{s_{\widehat{u}}} Z_j \sim \frac{\mathcal{N}(0,1)}{\sqrt{\chi^2_{n-k}/(n-k)}} = t_{n-k}. \tag{10.7}$$

Consequently, if we replace the unknown standard deviation $sd(\hat{\beta}_j|\boldsymbol{X})$ with the standard error $se_{hom}(\hat{\beta}_j)$ in the confidence interval formula, we have to replace the standard normal quantiles by t-quantiles:

$$I_{1-\alpha}^{(hom)} = [\hat{\beta}_j - t_{(1-\frac{\alpha}{2},n-k)} se_{hom}(\hat{\beta}_j);\ \hat{\beta}_j + t_{(1-\frac{\alpha}{2},n-k)} se_{hom}(\hat{\beta}_j)]$$

This interval is feasible and satisfies $P(\beta_j \in I_{1-\alpha}^{(hom)}) = 1-\alpha$ under (A1)–(A6).

Table 10.2: Student's t-distribution quantiles

df	0.9	0.95	0.975	0.99	0.995
1	3.08	6.31	12.71	31.82	63.66
2	1.89	2.92	4.30	6.96	9.92
3	1.64	2.35	3.18	4.54	5.84
4	1.53	2.13	2.78	3.75	4.60
5	1.48	2.02	2.57	3.36	4.03
6	1.44	1.94	2.45	3.14	3.71
8	1.40	1.86	2.31	2.90	3.36
10	1.37	1.81	2.23	2.76	3.17
15	1.34	1.75	2.13	2.60	2.95
20	1.33	1.72	2.09	2.53	2.85
25	1.32	1.71	2.06	2.49	2.79
30	1.31	1.70	2.04	2.46	2.75
40	1.30	1.68	2.02	2.42	2.70
50	1.30	1.68	2.01	2.40	2.68
60	1.30	1.67	2.00	2.39	2.66
80	1.29	1.66	1.99	2.37	2.64
100	1.29	1.66	1.98	2.36	2.63
$\to \infty$	1.28	1.64	1.96	2.33	2.58

We can use the `coefci` function from the `AER` package:

```r
library(AER)
coefci(fit)
```

```
                 2.5 %     97.5 %
(Intercept) -14.932204 -13.231372
education     2.898643   3.017705
female       -7.863490  -7.202643
```

```r
coefci(fit, level = 0.99)
```

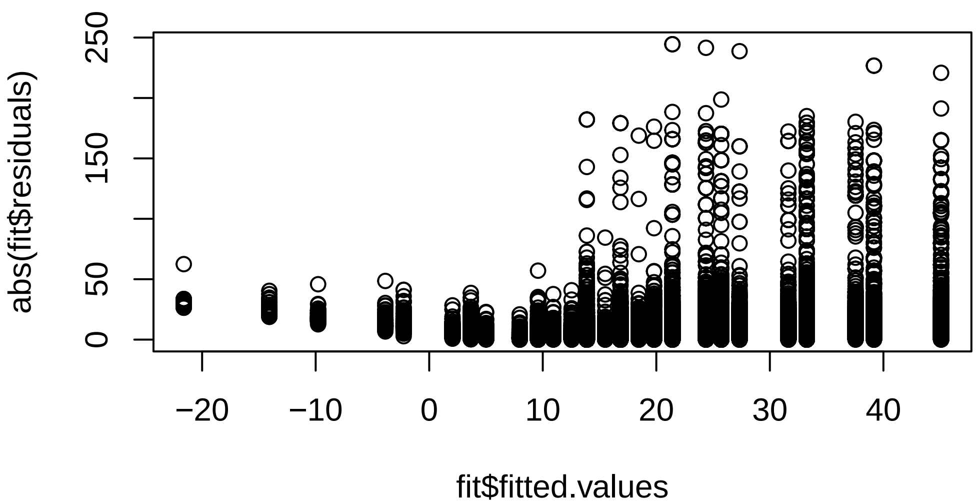
```
                 0.5 %     99.5 %
(Intercept) -15.199440 -12.964137
education     2.879936   3.036412
female       -7.967322  -7.098811
```

10.6 Confidence intervals: heteroskedasticity

The exact confidence interval $I_{1-\alpha}^{(hom)}$ is only valid under the restrictive assumption of homoskedasticity (A5) and normality (A6).

For historical reasons, statistics books often treat homoskedasticity as the standard case and heteroskedasticity as a special case. However, this does not reflect empirical practice since we have to expect heteroskedastic errors in most applications. It turns out that heteroskedasticity is not a problem as long as the robust standard errors are used.

```r
plot(abs(fit$residuals)~fit$fitted.values)
```



A plot of the absolute value of the residuals against the fitted values shows that individuals with predicted wages around 10 USD exhibit residuals with lower variance compared to those with higher predicted wage levels. Hence, the homoskedasticity assumption (A5) is implausible.

If (A5) does not hold, then standard deviation is

$$sd(\hat{\beta}_j|\boldsymbol{X}) = \sqrt{[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{D}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}]_{jj}}.$$

To estimate $sd(\hat{\beta}_j|\boldsymbol{X})$, we will have to replace the diagonal matrix

$$\boldsymbol{D} = diag(\sigma_1^2, \dots, \sigma_n^2)$$

by some sample counterpart

$$\widehat{\boldsymbol{D}} = diag(\hat{\sigma}_1^2, \dots, \hat{\sigma}_n^2).$$

Various **heteroskedasticity-consistent (HC)** standard errors have been proposed in the literature:

HC type	weights
HC0	$\hat{\sigma}_i^2 = \hat{u}_i^2$
HC1	$\hat{\sigma}_i^2 = \frac{n}{n-k}\hat{u}_i^2$
HC2	$\hat{\sigma}_i^2 = \frac{\hat{u}_i^2}{1-h_{ii}}$
HC3	$\hat{\sigma}_i^2 = \frac{\hat{u}_i^2}{(1-h_{ii})^2}$

HC0 replaces the unknown variances with squared residuals, and HC1 is a bias-corrected version of HC0. HC2 and HC3 use the leverage values h_{ii} (the diagonal entries of the influence matrix $\boldsymbol{P}$) and give less weight to influential observations.

HC1 and HC3 are the most common choices and can be written as

$$se_{hc1}(\hat{\beta}_j) = \sqrt{\Big[(\boldsymbol{X}'\boldsymbol{X})^{-1}\Big(\frac{n}{n-k}\sum_{i=1}^n \hat{u}_i^2 \boldsymbol{X}_i\boldsymbol{X}_i'\Big)(\boldsymbol{X}'\boldsymbol{X})^{-1}\Big]_{jj}},$$

$$se_{hc3}(\hat{\beta}_j) = \sqrt{\Big[(\boldsymbol{X}'\boldsymbol{X})^{-1}\Big(\sum_{i=1}^n \frac{\hat{u}_i^2}{(1-h_{ii})^2} \boldsymbol{X}_i\boldsymbol{X}_i'\Big)(\boldsymbol{X}'\boldsymbol{X})^{-1}\Big]_{jj}}.$$

All versions perform similarly well in large samples, but HC3 performs best in small samples and is the preferred choice.

HC standard errors are also known as **heteroskedasticity-robust standard errors** or simply **robust standard errors**.

Estimators for the full covariance matrix of $\hat{\boldsymbol{\beta}}$ have the form

$$\widehat{\boldsymbol{V}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\widehat{\boldsymbol{D}}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}.$$

The HC3 covariance estimator can be written as

$$\widehat{\boldsymbol{V}}_{hc3} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\Big(\sum_{i=1}^n \frac{\hat{u}_i^2}{(1-h_{ii})^2} \boldsymbol{X}_i\boldsymbol{X}_i'\Big)(\boldsymbol{X}'\boldsymbol{X})^{-1}.$$

Therefore, we can use confidence intervals of the form:

$$I_{1-\alpha}^{(hc)} = [\widehat{\beta}_j - t_{(1-\frac{\alpha}{2},n-k)} se_{hc}(\widehat{\beta}_j);\ \widehat{\beta}_j + t_{(1-\frac{\alpha}{2},n-k)} se_{hc}(\widehat{\beta}_j)].$$

In contrast to Equation 10.7, the distribution of the ratio $sd(\widehat{\beta}_j|\boldsymbol{X})/se_{hc}(\widehat{\beta}_j)$ is unknown in practice, and the t-statistic is not t-distributed.

However, for large n, we have

$$T_j^{(hc)} = \frac{\widehat{\beta}_j - \beta_j}{se_{hc}(\widehat{\beta}_j)} = \underbrace{\frac{sd(\widehat{\beta}_j|\boldsymbol{X})}{se_{hc}(\widehat{\beta}_j)}}_{\overset{p}{\to}1} \underbrace{Z_j}_{\sim\mathcal{N}(0,1)}$$

which implies that

$$\lim_{n\to\infty} P(\beta_j \in I_{1-\alpha}^{(hc)}) = 1-\alpha. \tag{10.8}$$

Therefore $I_{1-\alpha}^{(hc)}$ is an **asymptotic confidence interval** for β_j.

```r
## HC3 covariance matrix estimate Vhat-hc3
vcovHC(fit)
```

```
            (Intercept)    education       female
(Intercept)  0.25013606 -0.019590435  0.013394891
education   -0.01959043  0.001609169 -0.002173848
female       0.01339489 -0.002173848  0.026131235
```

```r
## HC3 standard errors
sqrt(diag(vcovHC(fit)))
```

```
(Intercept)   education      female 
 0.50013604  0.04011445  0.16165158 
```

```r
## HC1 standard errors
sqrt(diag(vcovHC(fit, type = "HC1")))
```

```
(Intercept)   education      female 
 0.50007811  0.04011017  0.16164436 
```

```r
coefci(fit, vcov = vcovHC, level = 0.99)
```

```
                 0.5 %     99.5 %
(Intercept) -15.370102 -12.793475
education     2.854842   3.061506
female       -7.949469  -7.116664
```

Robust confidence intervals can also be used and hold asymptotically under (A5). Therefore, the exact classical confidence intervals should only be used if there are very good reasons for the error terms to be homoskedastic and normally distributed.

10.7 Confidence interval with non-normal errors

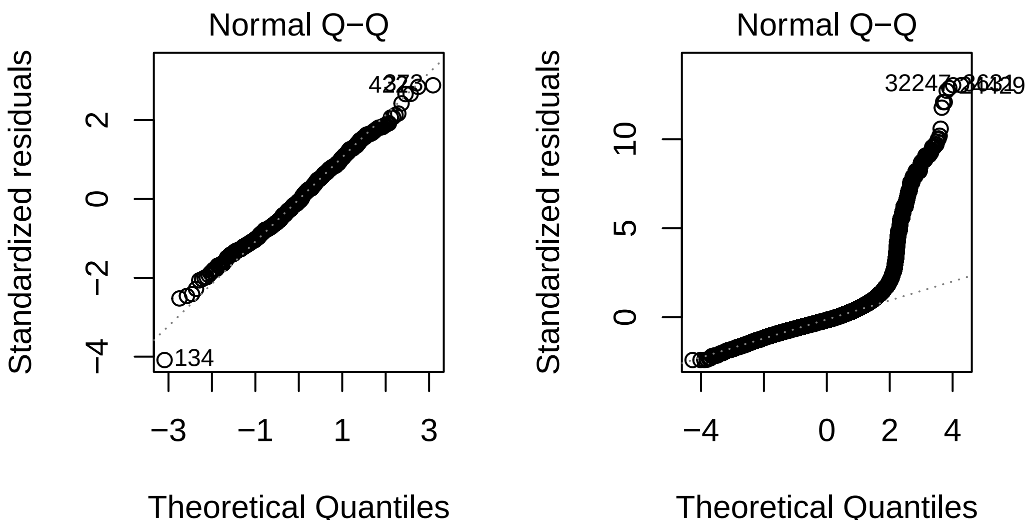
Similar to the homoskedasticity assumption (A5), the normality assumption (A6) is also not satisfied in most applications. A useful diagnostic plot is the Q-Q-plot.

The Q-Q-plot is a graphical tool to help us assess if the errors are conditionally normally distributed, i.e. whether assumption (A6) is satisfied.

Let $\hat{u}_{(i)}$ be the sorted residuals (i.e. $\hat{u}_{(1)} \leq ... \leq \hat{u}_{(n)}$). The Q-Q-plot plots the sorted residuals $\hat{u}_{(i)}$ against the $((i-0.5)/n)$-quantiles of the standard normal distribution.

If the residuals are lined well on the straight dashed line, there is indication that the distribution of the residuals is close to a normal distribution.

```r
par(mfrow = c(1,2))
# Normally distributed response variable
plot(lm(rnorm(500) ~ 1), which = 2)
plot(fit, which=2)
```



In the left plot you see the Q-Q-plot for an example with normally distributed errors. The right plot indicates that, in our regression of `wage` on `education` and `female`, the normality assumption is implausible.

If (A6) does not hold, then Z_j is not normally distributed, and it is unclear whether Equation 10.8 holds. However, by the central limit theorem, we still can establish that

$$\lim_{n\to\infty} P(\beta_j \in I_{1-\alpha}^{(hc)}) = 1 - \alpha.$$

Therefore, the robust confidence interval $I_{1-\alpha}^{(hc)}$ is asymptotically valid if (A1)–(A4) hold.

10.8 Central limit theorem

Convergence in distribution

Let $\boldsymbol{W}_n$ be a sequence of k-variate random variables and let $\boldsymbol{V}$ be a k-variate random variable

$\boldsymbol{W}_n$ **converges in distribution** to $\boldsymbol{V}$, written $\boldsymbol{W}_n \overset{d}{\to} \boldsymbol{V}$, if

$$\lim_{n\to\infty} P(\boldsymbol{W}_n \leq \boldsymbol{a}) = P(\boldsymbol{V} \leq \boldsymbol{a})$$

for all $\boldsymbol{a}$ at which the CDF of $\boldsymbol{V}$ is continuous.

If $\boldsymbol{V}$ has the distribution $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, we write $\boldsymbol{W}_n \overset{d}{\to} \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

Consider for simplicity the regression on an intercept only. In this case, we have $k = 1$ and $\widehat{\beta}_1 = \overline{Y}$ (see the second problem set).

By the univariate central limit theorem, the centered sample mean converges to a normal distribution:

Central Limit Theorem (CLT)

Let $\{Y_1, \dots, Y_n\}$ be an i.i.d. sample with $E[Y_i] = \mu$ and $0 < Var(Y_i) = \sigma^2 < \infty$. Then, the sample mean satisfies

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i - \mu\right) \xrightarrow{d} \mathcal{N}(0, \sigma^2).$$

Below, you will find an interactive shiny app for the central limit theorem:

SHINY APP: CLT

The same result can be extended to k-variate random vectors.

Multivatiate Central Limit Theorem (MCLT)

If $\{\boldsymbol{W}_1, \dots, \boldsymbol{W}_n\}$ is an i.i.d. sample with $E[\boldsymbol{W}_i] = \boldsymbol{\mu}$ and $Var(\boldsymbol{W}_i) = \boldsymbol{\Sigma} < \infty$. Then,

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{W}_i - \boldsymbol{\mu}\right) \overset{d}{\to} \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma})$$

(see, e.g., Stock and Watson Section 19.2).

If we apply the MCLT to the random sequence $\boldsymbol{W}_i = \boldsymbol{X}_i u_i$ with $E[\boldsymbol{X}_i u_i] = \boldsymbol{0}$ and $Var(\boldsymbol{X}_i u_i) = \boldsymbol{\Omega} = E[u_i^2 \boldsymbol{X}_i \boldsymbol{X}_i']$, then we get

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i u_i\right) \overset{d}{\to} \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Omega}).$$

Therefore, we get

$$\sqrt{n}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i \boldsymbol{X}_i'\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i u_i\right) \overset{d}{\to} \boldsymbol{Q}^{-1}\mathcal{N}(\boldsymbol{0}, \boldsymbol{\Omega}),$$

because $\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i \boldsymbol{X}_i' \overset{p}{\to} \boldsymbol{Q} = E[\boldsymbol{X}_i \boldsymbol{X}_i']$. Since $Var[\boldsymbol{Q}^{-1}\mathcal{N}(\boldsymbol{0}, \boldsymbol{\Omega})] = \boldsymbol{Q}^{-1}\boldsymbol{\Omega}\boldsymbol{Q}^{-1}$, we have the following central limit theorem for the OLS estimator:

Central Limit Theorem for OLS

Consider the general linear regression model Equation 10.1 under assumptions (A1)–(A4). Then, as $n \to \infty$,

$$\sqrt{n}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \overset{d}{\to} \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}^{-1}\boldsymbol{\Omega}\boldsymbol{Q}^{-1}).$$

A direct consequence is that the robust t-statistic is asymptotically standard normal:

$$T_j^{(hc)} = \frac{\widehat{\beta}_j - \beta_j}{se_{hc}(\widehat{\beta}_j)} \overset{d}{\rightarrow} \mathcal{N}(0,1).$$

Also note that the t-distribution t_{n-k} approaches the standard normal distribution as n grows. Therefore, we have

$$t_{n-k} \overset{d}{\rightarrow} \mathcal{N}(0,1)$$

and we can write

$$T_j^{(hc)} = \frac{\widehat{\beta}_j - \beta_j}{se_{hc}(\widehat{\beta}_j)} \overset{a}{\sim} t_{n-k}.$$

This notation means that $T_j^{(hc)}$ is asymptotically t-distributed. I.e., the distributions of $T_j^{(hc)}$ becomes closer to a t_{n-k} distribution as n grows.

Therefore, it is still reasonable to use t-quantiles in robust confidence intervals instead of standard normal quantiles. It also turns out that for smaller sample sizes, confidence intervals with t-quantiles tend to yield better small sample coverages that using standard normal quantiles.

10.9 CASchools data

Let's revisit the test score application from the previous section and compare HC-robust confidence intervals:

```
data(CASchools, package = "AER")
CASchools$STR = CASchools$students/CASchools$teachers
CASchools$score = (CASchools$read+CASchools$math)/2
fit1 = lm(score ~ STR, data = CASchools)
fit2 = lm(score ~ STR + english, data = CASchools)
fit3 = lm(score ~ STR + english + lunch, data = CASchools)
fit4 = lm(score ~ STR + english + lunch + expenditure, data = CASchools)
library(stargazer)
```

```
coefci(fit1, vcov=vcovHC)
```

```
                 2.5 %   97.5 %
(Intercept) 678.371140 719.4948
STR          -3.310516  -1.2491
```

```r
coefci(fit2, vcov=vcovHC)
```

```
                 2.5 %      97.5 %
(Intercept) 668.7102930 703.3541961
STR          -1.9604231  -0.2421682
english      -0.7112962  -0.5882574
```

```r
coefci(fit3, vcov=vcovHC)
```

```
                 2.5 %      97.5 %
(Intercept) 689.0614539 711.2384604
STR          -1.5364346  -0.4601833
english      -0.1869188  -0.0562281
lunch        -0.5951529  -0.4995380
```

The confidence intervals for `STR` in the first three models do not cover 0 and are strictly negative. This gives strong statistical evidence that the marginal effect of `STR` on `score` is negative, holding `english` and `lunch` fixed.

```r
coefci(fit4, vcov=vcovHC)
```

```
                   2.5 %       97.5 %
(Intercept) 645.329067184 686.64732942
STR          -0.882408250   0.41163186
english      -0.192981575  -0.06370184
lunch        -0.592410029  -0.50037547
expenditure   0.001738419   0.00550568
```

In the fourth model, the point estimator for the marginal effect of `STR` is negative, but the confidence interval also covers positive values. Therefore, there is no statistical evidence that the marginal effect of `STR` on `score` holding `english`, `lunch`, and `expenditure` fixed.

However, as discussed in the previous section, `expenditure` is a bad control for `STR` and should not be used to estimate the effect of class size on test score.

10.10 R-codes

statistics-sec10.R